

Emerging criticality in heteroclinic dynamics

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Like phase transitions, bifurcations mark borderlines between different behavior, and as borderlines they are predestinated for displaying emergent new behavior, sometimes enabling dynamics that is rich, versatile and functionally flexible, involving features from the phases on both sides of the transition. Universal features at bifurcation points arise when the singularities are approached in the infinite-time limit, while they arise at critical points in the large-volume (thermodynamic) limit for phase transitions. In thermodynamic phase transitions the divergence in the thermodynamic limit is announced in a characteristic way for large but finite volumes. This is revealed in a finite-size scaling analysis. In analogy, a finite-time scaling analysis for local bifurcations shows characteristic scaling behavior depending of the type of bifurcations [1-2]. Slow transient behavior plays then a similar role as critical slowing down at a second-order phase transition in the vicinity of a critical point there.

Being poised at criticality may be one of the guiding organization principles in living systems, as operating near criticality has certain functional advantages. One of the trademarks is the divergence of the response or susceptibility, which is useful for biological sensing systems. The emergence of long correlations in space and time may be exploited for coordinated behavior, and together with critical slowing down it may favor slowly decaying memories. The dynamical repertoire, in particular, the variability of possible spatiotemporal patterns is maximal at criticality. Thus, the typical features of criticality are a proliferation of the dynamical repertoire, critical slowing down of the dynamics and a high sensitivity to perturbations in the vicinity of a critical parameter value. As these features support the storage, transformation and processing of information, it is natural to search for applications to brain dynamics.

Criticality in the more specific sense of being at the edge of synchrony allows the largest diversity of synchronization patterns and scale-free avalanches of activity of neural systems. Actually, at the edge of synchrony, both types of critical points, those at phase transitions and at individual bifurcation points, are observed in the auditory system of vertebrates. In the simplest formulation, a single hair cell as the ear's sensory receptor is well described as a Stuart-Landau oscillator that corresponds to the normal form of a Hopf bifurcation. At the very bifurcation point that separates fixed-point (damped oscillatory) from oscillatory behavior, the response-to-noise ratio is maximal and provides a very efficient selection of external frequencies along with an amplification mechanism. When many such Stuart-Landau oscillators are coupled in the cochlea of the vertebrate ear, a true phase transition emerges with a critical point that turns out to be the optimal working point of the system [3-5].

Similarly, Hopf bifurcations provide the optimal working points for the temporal dynamics of resting state fluctuations in the brain when analyzed in a whole brain neural mass model. Again, the local dynamics is chosen as the normal form of a Hopf bifurcation, so the network model amounts to a coupled system of Stuart-Landau oscillators, here also exposed to noise. The coupling matrix is determined by the connectivity structure between brain areas, fitted to empirical functional connectivity dynamics. As it turns out, the optimal working point in the sense that its parameter choice fits best the characteristics of the empirical functional magnetic resonance imaging data is again at the edge between asynchronous low activity and collective oscillations on the other side [6].

As an alternative to coupled oscillatory units like Stuart-Landau oscillators, a more formalized mathematical framework to describe metastable dynamics with pronounced dwell times in the metastable states is provided by heteroclinic dynamics, considered in our work [7]. In heteroclinic dynamics, the system evolves from one metastable state to the next in a reproducible and ordered way. One possible realization is via winnerless competition dynamics, formulated in terms of generalized Lotka-Volterra equations that predict whole sequences of discrete metastable states. In

the simplest case, these metastable states correspond to saddle equilibria of the dynamics. In relation to cognitive dynamics, in the vicinity of a saddle, a certain information item is dominant and the temporary winner in the game of competition. The saddles are connected via heteroclinic orbits. The heteroclinic orbits may form short sequences, cycles or entire heteroclinic networks, which are networks in phase space. We consider various heteroclinic networks. While in previous analyses we located the bifurcation points in phase space only to explore the different dynamical regimes in the “bulk” of the respective regimes (that is, for parameters not close to the bifurcation points), we explore in this work the dynamics that is observed at the very bifurcation points and their immediate vicinity. We want to analyze the conditions under which bifurcation points in heteroclinic dynamics also provide suitable working points for brain dynamics, sharing features of criticality.

We first consider a May-Leonard system, one of the simplest systems that exhibits a heteroclinic cycle. The system corresponds to a mean-field version of a rock-paper-scissors game with cyclic competition that is winnerless in the sense that the winners change in the course of time. For a certain combination of parameters, we find a degenerate Hopf bifurcation, at which the three-species coexistence equilibrium gets stable, while the heteroclinic cycle loses its attraction. Precisely at this bifurcation point, the system develops two conserved quantities which turn out to be the sum of item concentrations (in a continuous formulation) and the product of concentrations of information items. The dynamics forces the trajectory to approach a plane P . If the initial conditions are chosen outside this plane, the dynamics is attracted fast toward P , and then remains close to P , where it is slow. To each possible value of the product belongs one periodic orbit (with specific amplitude), and all initial conditions corresponding to that specific product value lead to the same periodic orbit. This means that the system strongly depends on the initial conditions as they fix the orbit. This leads to a proliferation of states, differing in the amplitudes of the periodic orbits that emerge at the bifurcation point. Further, in the immediate vicinity of this bifurcation point, we observe a critical regime with slowing down of the dynamics over the entire plane and scaling of the decay time of the periodic orbits and the product over the item concentrations.

As a second heteroclinic network we consider a hierarchical one with a hierarchy in time scales, corresponding to slow oscillations modulating fast oscillations. The analysis can be partially traced back to the former case. Here we focus on a bifurcation point at which the hierarchy is reduced from two levels (a heteroclinic cycle of heteroclinic cycles) to one level; the former hierarchical heteroclinic network becomes a heteroclinic cycle between three-species saddle equilibria. As a novel feature at this bifurcation point, we observe the emergence of a heteroclinic cycle between three manifolds that are densely covered by periodic orbits. Again, which periodic orbit is selected from the manifold when the trajectory comes close to it, sensitively depends on the initial conditions due to the hidden conservation laws. Here we also observe a critical regime with slow decay of periodic orbits, scaling in the sense that the time evolution of the products of item concentrations (spanning the different subspaces) collapse to a master curve, while such characteristic precursors of the dynamics at the bifurcation point vanish outside the critical regime. This means that the observed features of criticality at the considered bifurcation points qualify these points as candidates for working points in systems with heteroclinic dynamics which store and transfer information. From a more general perspective, our results add to the observation that emergent features may arise at borderlines. Here these borderlines separate different dynamical regimes in heteroclinic dynamics.

As in case of other established working points such as Hopf bifurcations in Stuart-Landau oscillators, it remains open how the brain might tune the dynamics toward these working points. However, it will be interesting to see whether the kind of dynamical criticality observed here can be exploited when heteroclinic dynamics is used for computation in artificial systems with the option of actively tuning the parameters to such bifurcation points. In particular, as a future direction of this work one may study stochastic versions of heteroclinic dynamics, formulated in terms of predation-prey rates, birth, death and diffusion rates and a master equation that determines the probability to find n items

(species) at time t of type i at site j on a spatial grid. Tuning the bifurcation parameter according to different protocols, it will be interesting to check the behavior of the Fisher information metric [8-10], here across the bifurcation point. At the bifurcation point, the dynamics changes, for example, between the stochastic counterparts of a collective fixed-point and synchronized heteroclinic cycles with an infinite set of periodic orbits at the bifurcation point. It should be noticed that our signatures of criticality do not emerge in an order-disorder transition in the deterministic limit that we considered so far, but between motions with a different degree of coherence when coupled on a spatial grid.

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