Self-organisation and thermodynamic efficiency: magnets, swarms, cities and contagions...

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Typically, self-organisation is defined as a spontaneous formation of spatiotemporal structures or functions in a system containing multiple interacting components, in the absence of specific external interference [1]. To explain this process, Haken contrasted *order parameters* and *control parameters* [1]: slow variations of a suitable control parameter may induce an abrupt change, a *phase transition*, in an observable order parameter.

We review several studies of self-organization as a thermodynamic phenomenon in the presence of changes in the generalised internal energy and the generalised work carried out on, or extracted from, the system. The studies include: (i) Grégoire and Chaté model of collective motion [2, 3], (ii) Boltzmann-Lotka-Volterra dynamics of spatial urban agglomeration [4, 5], (iii) SIS epidemic diffusion on Watts–Strogatz random graphs [6, 7], and (iv) the Curie-Weiss model of ferromagnetism [8, 9]. In the context of the first law of thermodynamics, we trace changes in relevant thermodynamic quantities, and observe that a variation of a relevant control parameter drives the system across a phase transition.

These results identify critical regimes and show that during the phase transition, where the configuration entropy S of the system decreases, the rates of change of the work and of the internal energy also decrease. Importantly, the reduction of entropy achieved through the expenditure of generalised work W_{gen} is shown to peak at phase transitions. We relate this to the thermodynamic efficiency defined as a ratio of the gained predictability (that is, the increase in the internal order) to the amount of work required to change the underlying control parameter θ:

$$\eta \equiv \frac{-dS/d\theta}{d\langle \beta W_{gen} \rangle / d\theta} = \frac{-dS/d\theta}{\int_{\theta}^{\theta^*} F(\theta') d\theta'} \;, \tag{1}$$

where β is the inverse temperature. The second equation interprets the thermodynamic efficiency η in purely computational terms, as the ratio of reduced uncertainty (defined via Shannon entropy), obtained at θ, to the Fisher information $F(\theta)$ accumulated on a trajectory between the current state θ and the state of perfect order θ^*. The latter is identified by the zero-response point θ^*, i.e., the point for which small changes incur no work. The Fisher information $F(\theta) = F_X(\theta)$ measures how sensitive a distribution $P(x;\theta)$ of an observable random variable X is to changes in parameter θ on average: $F_X(\theta) = E\left[\left(\frac{\partial}{\partial\theta} \log P(x;\theta)\right)^2\right]$. Its integration between θ and θ^* captures the overall sensitivity of the computational process along this trajectory [3]. This computational interpretation holds under a quasi-static protocol for which the first law of thermodynamics yields:

$$F(\theta) = -\frac{d^2\langle \beta W_{gen} \rangle}{d\theta^2} \;. \tag{2}$$

Initially, the thermodynamic efficiency (1) was proposed to characterise collective motion of self-propelled particles (e.g., flocks, swarms, active matter) [3]. It was demonstrated that η maximises at the induced kinetic phase transition [2] where the swarm transitions from disordered to coherent motion (Fig. 1), due to quasi-static changes either in the particles' *alignment strength* or the *number of interacting neighbours* [3].

The thermodynamic efficiency was also found to peak during urban transformations [5], driven by quasi-static changes in the *social disposition*: a control parameter measuring the attractiveness of different urban areas. In terms of urban dynamics [4], the efficiency η expressed the ratio of the gained predictability of income flows to the amount of work required to change the social disposition. The efficiency was shown to peak at a phase transition separating dispersed and polycentric phases of urban dynamics (Fig. 1) [5].

Harding et al. [7] investigated the thermodynamic efficiency of quasi-static epidemic processes diffusing on Watts–Strogatz random graphs [6], defined for a value of some control parameter (e.g., the *infection transmission rate*). The efficiency η was again defined as the ratio of the reduction in uncertainty to the expenditure of work needed to change the control parameter. This was interpreted from two perspectives: (a) the efficiency of an intervention process consuming work needed to reduce the transmission rate, and (b) the efficiency of the pathogen emergence — a process which increases the transmission rate while extracting the work from the environment. Importantly, the efficiency was shown to peak at the epidemic threshold (Fig. 1) [7].

*Acknowledgments: Emanuele Crosato, Nathan Harding, Richard E. Spinney, and Ramil Nigmatullin.

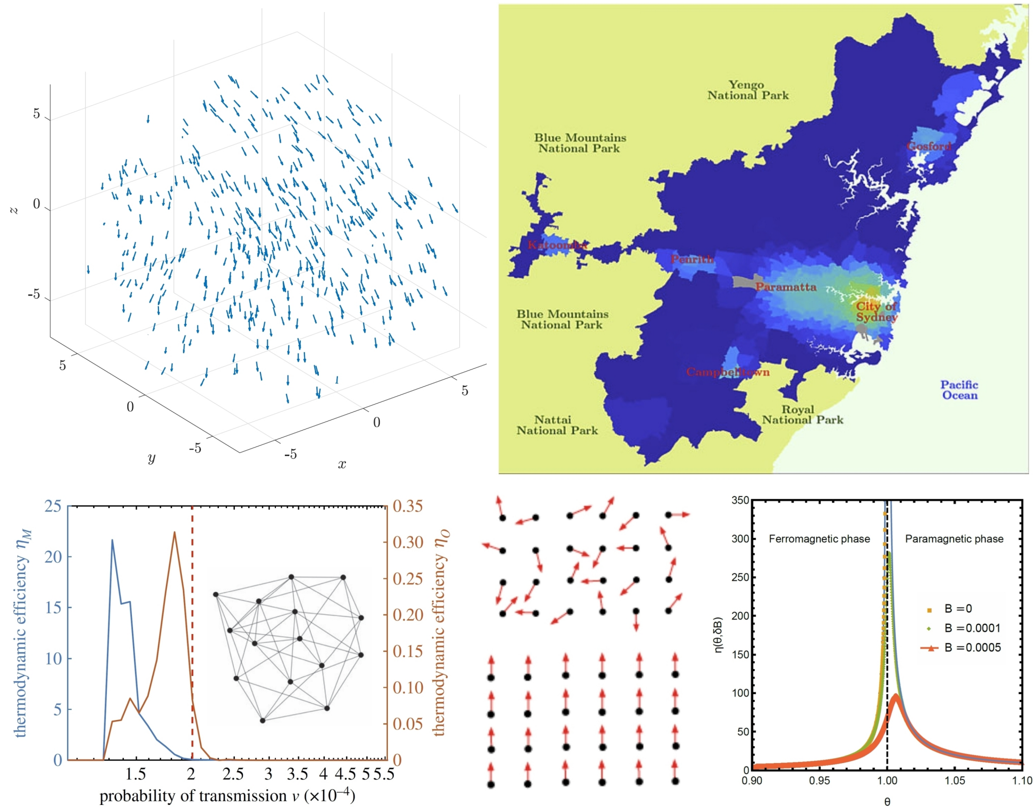

Figure 1: Phase transitions in self-organising systems: (top-left) swarming behaviour of self-propelled particles [3]; (top-right) polycentricity in urban transformations [5]; (bottom-left) efficiency η of epidemic diffusion on Watts–Strogatz random graphs maximises at an epidemic threshold [7]; (bottom-right) efficiency η at paramagnetic to ferromagnetic phase transition [9, 10].

Finally, the thermodynamic efficiency of interactions was defined and analytically derived for the exactly solvable Curie-Weiss (fully connected) Ising model [8, 9]. It was shown that the efficiency η diverges at the critical point of a second order phase transition during quasi-static perturbations in control parameters: either the *external field* or the *coupling strength* (Fig. 1).

Thus, the efficiency of self-organisation is generally expected to diverge (maximise in finite-size systems) at a second-order phase transition, rather than at the macroscopically stable low-dimensional phases. These results may not only explain a range of critical phenomena across physical, biological and social domains, but also help in forming a general principle for guiding self-organising dynamics towards their most efficient regimes.

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